Journal of Engineering Mathematics **24**: 25–43, 1990.
© 1990 *Kluwer Academic Publishers. Printed in the Netherlands.*

Exact solutions for vertical drainage and redistribution in soils

P. BROADBRIDGE[1] and C. ROGERS[2]

[1]*Mathematics Department, La Trobe University, Bundoora 3083, Victoria, Australia*; [2]*Department of Mathematical Sciences, Loughborough University of Technology, Loughborough LE113TU, United Kingdom*

Received 10 January 1989; accepted 6 March 1989

Abstract. We solve a versatile nonlinear convection-diffusion model for nonhysteretic redistribution of liquid in a finite vertical unsaturated porous column. With zero-flux boundary conditions, the nonlinear boundary-value problem may be transformed to a linear problem which is exactly solvable by the method of Laplace transforms. In principle, this technique applies to arbitrary initial conditions.

The analytic solution for drainage in an initially saturated semi-finite column is compared to previously available approximate analytic solutions, obtained by assuming constant diffusivity, as in the Burgers equation, or by neglecting diffusivity, as in the hyperbolic model. Contrary to popular opinion, the hyperbolic model has more than one shock-free solution in a semi-infinite medium $z \geqslant 0$, as opposed to an infinite medium $z \in \mathbb{R}$. However, both the Burgers equation and an improved hyperbolic model underestimate diffusivity at high liquid contents and consequently overestimate the curvature of the soil liquid content profile.

1. Introduction

The redistribution of soil water is an important natural process in ecology, agriculture and water-resources management. In addition, controlled redistribution has been used by Nielsen et al. [1], Mahmoodian-Shooshtari et al. [2] and Jones and Wagenet [3] to measure soil hydraulic properties. Such measurements rely upon the assumed dynamical theory of unsaturated flow during drainage.

One-dimensional vertical flows must obey the equation of continuity

$$\frac{\partial \theta}{\partial t} + \frac{\partial v}{\partial z} = 0\,, \tag{1}$$

where $\theta(z, t)$ is the volumetric moisture fraction, v is the associated flux, z is depth and t is time. During redistribution, the upper boundary condition is

$$v = 0\,, \qquad z = 0\,. \tag{2}$$

From the form of Darcy's law given by Childs [4] to account for unsaturated flow with hysteresis, the flux is given by

$$v = -K(\theta)\left[\frac{\partial \Psi(\theta, \theta_*)}{\partial \theta}\,\frac{\partial \theta}{\partial z} + \frac{\partial \Psi(\theta, \theta_*)}{\partial \theta_*}\,\frac{\mathrm{d}\theta_*}{\mathrm{d}z} - 1\right], \tag{3}$$

where K is the hydraulic conductivity, Ψ is the (negative) capillary potential, or nongravitational potential energy per unit weight, and θ_* is the moisture content at which drying takes over from wetting.

A particular example of the redistribution problem is the deep-drainage problem, in which an initially saturated ($\theta_i = \theta_s$), notionally semi-infinite ($l = \infty$) column is allowed to drain. In this circumstance, moisture decreases monotonically everywhere and hysteresis effects are ignorable. Equations (1) and (3) then imply the nonhysteretic unsaturated-flow equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D(\theta)\,\frac{\partial \theta}{\partial z}\right) - \frac{\mathrm{d}K}{\mathrm{d}\theta}\,\frac{\partial \theta}{\partial z}\,, \tag{4}$$

where D is the soil water diffusivity,

$$D(\theta) = K(\theta)\,\frac{\partial \Psi(\theta, \theta_s)}{\partial \theta}\,. \tag{5}$$

When more general initial conditions are imposed, one expects distinct wetting and drying zones and the effect of hysteresis may be considerable, as shown, for example, by Youngs and Poulovassilis [5] and by Watson and Sardana [6]. However, due to the difficulty of the problem, hysteresis has not been accounted for in exact analytic models. It has been incorporated in approximate analytic or numerical models by Whisler and Klute [7], Rubin [8], Staple [9], Peck [10] and Perrens and Watson [11].

Even when hysteresis has been neglected, analytic models have continued to suffer from the deficiencies summarized by Gardner et al. [12]. The relative magnitudes of so-called gravitational and diffusive components of the soil water flux were investigated by Talsma [13]. In some, but not all, zones of the soil, one may neglect either the diffusive contribution or the gravitational contribution to the flux gradient (i.e. respectively the first and second terms in the right-hand side of (4)) and thereby obtain previously used approximate analytic models. For particular soil hydraulic functions $D(\theta)$ and $K(\theta)$, separation of time and space variables in equation (4) is possible but Raats [14] pointed out that a solution of the form $\theta(z, t) = f(z)g(t)$ is feasible only for a limited class of initial conditions.

As discussed by Philip [15], if the functions $D(\theta)$ and $K(\theta)$ represent measured properties of real soils, then both the quasilinear flow equation (4) and the boundary condition (2) are, in fact, highly nonlinear. However, for various choices of $D(\theta)$ and $K(\theta)$ appropriate to other applications, Fokas and Yortsos [16], Rosen [17] and Rogers et al. [18] have been able to transform the nonlinear equation (4), subject to constant-flux boundary conditions, to an exactly solvable linear problem. A similar model solvable equation of the form (4) has been shown by Broadbridge and White [19] and White and Broadbridge [20] to be well-suited to unsaturated flow in soils. Here we use this model to solve the nonhysteretic redistribution problem exactly. This enables us to better assess the effect of various additional popular assumptions, such as the neglect of the diffusive contribution to the soil water flux. In principle, our technique applies to general initial conditions.

2. Analytic solution (semi-infinite column)

Recently Broadbridge and White [19] and White and Broadbridge [20] made use of the unsaturated-flow equation (4) with

$$K = K_s\Theta^2(C-1)/(C-\Theta) \tag{6a}$$

and

$$D = h(C)S^2(\theta_s - \theta_n)^{-2}(C-\Theta)^{-2}\,. \tag{6b}$$

Here Θ is the normalized water content, $\Theta = (\theta - \theta_n)/(\theta_s - \theta_n)$, where θ_n is the initial uniform low volumetric moisture content, θ_s is the moisture content at saturation, S is the sorptivity defined by Philip [21] and K_s is the value of conductivity at saturation $\theta = \theta_s$. For a wide variety of real soils, the representation (6a–b) is reasonable and the parameter C between 1.02 and 1.5 (White and Broadbridge [20]).

The function $h(C)$ is the inverse of

$$C(h) = (4h/\pi)^{1/2} \exp(-1/4h)/\operatorname{erfc}(\tfrac{1}{2}h^{-1/2}) . \tag{7}$$

In terms of dimensionless variables, our version of the nonlinear convection-diffusion equation (4) is

$$\frac{\partial \Theta}{\partial t_*} = \frac{\partial}{\partial z_*}\left[D_* \frac{\partial \Theta}{\partial z_*}\right] - \frac{\mathrm{d}K_*}{\mathrm{d}\Theta}\frac{\partial \Theta}{\partial z_*} , \tag{8}$$

where

$$z_* = z/\lambda_s , \qquad t_* = t/t_s , \qquad D_* = C(C-1)/(C-\Theta)^2 \tag{9}$$

and

$$K_* = (C-1)\Theta^2/(C-\Theta) . \tag{10}$$

The length and time scales are

$$\lambda_s = h(C)S^2/C(C-1)(\theta_s - \theta_n)K_s \tag{11}$$

and

$$t_s = h(C)S^2/C(C-1)K_s^2 . \tag{12}$$

The factor $h(C)/C(C-1)$ is of the order of unity, since $h(C)/C(C-1) \to \frac{1}{2}$, $\pi/4$ as $C \to 1^+, \infty$. λ_s is a natural unit for capillary rise, an intrinsic property of the medium. The dimensionless water-flux density is

$$v_* = K_* - D_* \, \partial\Theta/\partial z_* . \tag{13}$$

When $D_*(\Theta)$ and $K_*(\Theta)$ take the form of (9–10), equation (8) is exactly solvable when subjected to constant-flux boundary conditions. The redistribution problem, involving a zero-flux boundary condition,

$$v_* = 0 , \qquad z_* = 0 , \tag{14}$$

is a particular case of the solvable system.

However, the mathematical problem now being addressed is that the initial conditions $\Theta = \Theta_i(z_*)$ may be more complicated than those assumed by Broadbridge and White [19]. For example, in some experiments (e.g. Talsma [13] and Mahmoodian-Shooshtari et al. [2]), a prescribed volume V_0 of water is rapidly injected into a soil column of cross-sectional area

A, after which redistribution is monitored. In such cases, the ideal initial conditions would be

$$\Theta = \Theta_i(z_*) = 1\,, \qquad z < l_* = V_0/[A(\theta_s - \theta_n)\lambda_s]\,, \\ = 0\,, \qquad z > l_*\,. \tag{15}$$

We shall derive an exact solution for these initial conditions but the same technique applies to general initial conditions.

The Kirchhoff [22] transformation

$$\mu = \int_{-\infty}^{\Theta} D_*(\omega)\,\mathrm{d}\omega \tag{16}$$

$$= C(C-1)/(C-\Theta)\,, \tag{17}$$

modifies (8) to

$$\frac{1}{D_*}\frac{\partial \mu}{\partial t_*} = \frac{\partial^2 \mu}{\partial z_*^2} - \frac{\mathrm{d}K_*}{\mathrm{d}\mu}\frac{\partial \mu}{\partial z_*}\,. \tag{18}$$

The boundary and initial conditions are

$$v_* = K_* - \frac{\partial \mu}{\partial z_*} = 0\,, \qquad z_* = 0 \qquad (K_* = C\mu^{-1}[\mu - (C-1)]^2)\,, \tag{19a}$$

$$\mu \to C - 1 \quad \text{as} \quad z_* \to \infty\,, \tag{19b}$$

and

$$\mu = \begin{cases} C\,, & z_* < l_*\,, \quad t_* = 0\,, \\ C-1\,, & z_* > l_*\,, \quad t_* = 0\,. \end{cases} \tag{20}$$

The Storm [23] transformation

$$Z = \int_0^{z_*} D_*^{-1/2}(\mu(z_*, t_*))\,\mathrm{d}z_* = [C(C-1)]^{1/2}\int_0^{z_*} \mu^{-1}\,\mathrm{d}z_* \tag{21a}$$

and

$$T = t_* \tag{21b}$$

then changes (18) to

$$\frac{\partial \mu(Z,T)}{\partial T} = \frac{\partial^2 \mu}{\partial Z^2} + m^{1/2}[1 - \mu/(C-1)]\frac{\partial \mu}{\partial Z}\,, \tag{22}$$

where

$$m = 4C(C-1)\,. \tag{23}$$

The boundary and initial conditions (19) and (20) are equivalent to

$$[\mu - (C-1)]^2 - \left[\frac{C-1}{C}\right]^{1/2} \frac{\partial \mu}{\partial Z} = 0\,, \quad Z = 0\,, \tag{24a}$$

$$\mu \rightarrow C - 1 \quad \text{as} \quad Z \rightarrow \infty\,, \tag{24b}$$

and

$$\mu = \begin{cases} C\,, & Z < L = [(C-1)/C]^{1/2} l_*\,, \quad T = 0\,, \\ C - 1\,, & Z > L\,, \qquad\qquad\qquad\qquad T = 0\,. \end{cases} \tag{25}$$

Equation (22) is essentially Burgers' equation. Following Forsyth [24], Hopf [25] and Cole [26], after an application of the transformation

$$m^{1/2}[1 - \mu/(C-1)] = \frac{2}{u}\frac{\partial u}{\partial Z}\,, \tag{26}$$

(22) can be replaced by the linear diffusion equation

$$\frac{\partial u}{\partial T} - \frac{\partial^2 u}{\partial Z^2} = 0\,, \tag{27}$$

subject to upper boundary condition $\partial^2 u/\partial Z^2 = 0$, which is equivalent to $\partial u/\partial T = 0$, by (27). From (26), u is defined up to a free gauge transformation $u(Z, t) \rightarrow \alpha u(Z, t)$ (α constant) and we can satisfy the boundary and initial conditions (24)–(25) by imposing

$$u = 1\,, \quad Z = 0\,, \tag{28a}$$

$$u \rightarrow \exp(-l_*) \quad \text{as} \quad Z \rightarrow \infty\,, \tag{28b}$$

$$u = \begin{cases} \exp(-[C/(C-1)]^{1/2} Z]\,, & Z \leqslant L\,, \quad T = 0\,, \\ \exp(-l_*)(\text{constant})\,, & Z \geqslant L\,, \quad T = 0\,. \end{cases} \tag{29}$$

The linear problem (27)–(29) may be solved by the method of Laplace transforms (see Appendix), resulting in

$$\begin{aligned} Z \leqslant L: \quad u = {} & \exp(-MZ + M^2T) + \operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2}) \\ & + \tfrac{1}{2}\exp(MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}[L + Z]T^{-1/2} + MT^{1/2}) \\ & - \tfrac{1}{2}\exp(-MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}[L - Z]T^{-1/2} + MT^{1/2}) \\ & - \tfrac{1}{2}\exp(-ML)\operatorname{erfc}(\tfrac{1}{2}[L + Z]T^{-1/2}) \\ & + \tfrac{1}{2}\exp(-ML)\operatorname{erfc}(\tfrac{1}{2}[L - Z]T^{-1/2}) \\ & - \tfrac{1}{2}\exp(-MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2} - MT^{1/2}) \\ & - \tfrac{1}{2}\exp(MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2} + MT^{1/2})\,, \end{aligned} \tag{30a}$$

$$Z \geq L: \quad u = \exp(-ML) + \operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2})$$
$$- \tfrac{1}{2}\exp(MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2} + MT^{1/2})$$
$$- \tfrac{1}{2}\exp(-MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2} - MT^{1/2})$$
$$+ \tfrac{1}{2}\exp(MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}[Z + L]T^{-1/2} + MT^{1/2})$$
$$+ \tfrac{1}{2}\exp(-MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}[Z - L]T^{-1/2} - MT^{1/2})$$
$$- \tfrac{1}{2}\exp(-ML)\operatorname{erfc}(\tfrac{1}{2}[Z + L]T^{-1/2})$$
$$- \tfrac{1}{2}\exp(-ML)\operatorname{erfc}(\tfrac{1}{2}[Z - L]T^{-1/2}), \tag{30b}$$

where

$$M = \left[\frac{C}{C-1}\right]^{1/2}. \tag{31}$$

To obtain the matrix flux potential μ via (26), we use

$$Z \leq L: \quad \frac{\partial u}{\partial Z} = -M\exp(-MZ + M^2T)$$
$$+ \tfrac{1}{2}M\exp(MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}[L + Z]T^{-1/2} + MT^{1/2})$$
$$+ \tfrac{1}{2}M\exp(-MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}[L - Z]T^{-1/2} + MT^{1/2})$$
$$+ \tfrac{1}{2}M\exp(-MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2} - MT^{1/2})$$
$$- \tfrac{1}{2}M\exp(MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2} + MT^{1/2}), \tag{32a}$$

$$Z \geq L: \quad \frac{\partial u}{\partial Z} = \tfrac{1}{2}M\exp(MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}[Z + L]T^{-1/2} + MT^{1/2})$$
$$- \tfrac{1}{2}M\exp(-MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}[Z - L]T^{-1/2} - MT^{1/2})$$
$$+ \tfrac{1}{2}M\exp(-MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2} - MT^{1/2})$$
$$- \tfrac{1}{2}M\exp(MZ + M^2T)\operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2} + MT^{1/2}). \tag{32b}$$

For a fixed non-negative value of Z, Θ is given by

$$\Theta = C\left[1 - \left(1 - 2m^{-1/2}u^{-1}\frac{\partial u}{\partial Z}\right)^{-1}\right]. \tag{33}$$

The depth z_* is obtained by inverting (21a),

$$z_* = \int_0^Z D_*^{1/2}\,dZ$$
$$= C^{-1}[\tfrac{1}{2}m^{1/2}Z - \ln u], \quad \text{using (28a)}. \tag{34}$$

The expressions (30)–(34) provide an exact parametric solution, requiring no numerical integration.

In the deep-drainage problem, the whole column is initially saturated ($L = \infty$), all terms in (30) involving L become negligible and the solution agrees with that of Broadbridge and

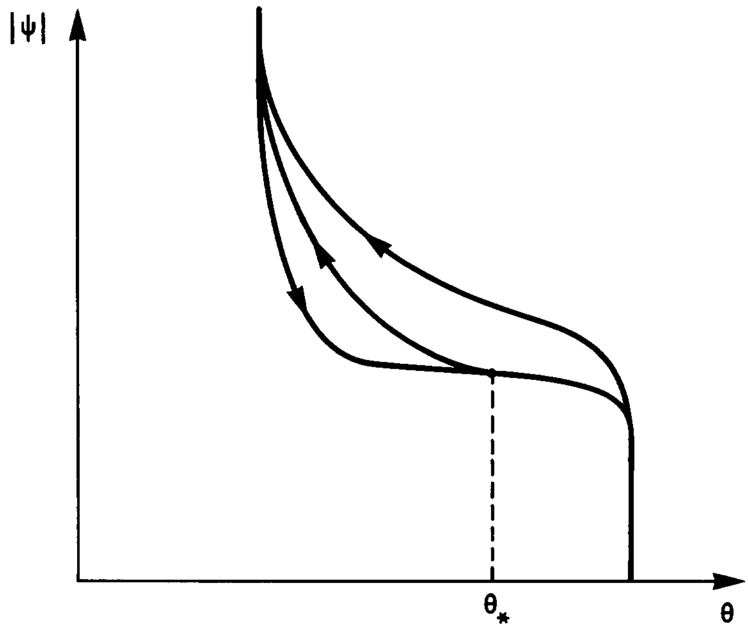

Fig. 1. Hysteresis curves in the moisture characteristic of a typical soil, including boundary-drying curve, primary-drying curve originating at θ_*, and boundary-wetting curve.

White [19] for the special case of zero flux $R = 0$ at $z = 0$, except that the initial moisture content is now θ_s rather than θ_n. From (33), we obtain an explicit expression for the surface moisture content during deep drainage,

$$\Theta = C - [1 + (1/[C-1] - 1)\, e^{M^2 t/t_s}\, \mathrm{erfc}(M[t/t_s]^{1/2})]^{-1} \quad \text{at} \quad z = 0 . \tag{35}$$

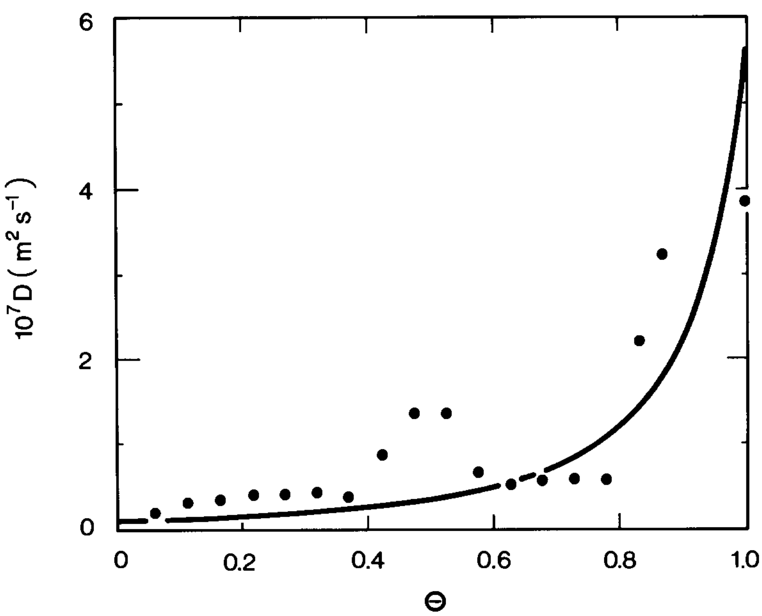

Fig. 2. Soil water diffusivity, D, in drying Yolo light clay. ● estimated from measurements of $K(\theta)$ and $\Psi(\theta)$ (Moore [27]). —— current analytic model.

An appropriate time scale for surface drainage is

$$M^{-2}t_s = C^{-2}h(C)S^2K_s^{-2}\,. \tag{36}$$

For Yolo light clay with $\theta_n = 0.2376$ and $\theta_s = 0.4950$, we take $C = 1.17$ and $K_s = 1.23 \times 10^{-7}$ m s^{-1}. The raw conductivity data and fitted expression (6a) are given in Fig. 3a of White and Broadbridge [20]. In a theory of deep drainage, the appropriate diffusivity (5) is related to the derivative $d\Psi/d\theta$ taken along the boundary drying moisture characteristic whereas in a theory of infiltration, the derivative is taken along the boundary wetting characteristic (Fig. 1). Using the drying-moisture characteristic data measured by Moore [27], we calculated the sorptivity S by the method of Philip and Knight [28] to be 0.987×10^{-4} m s$^{-1/2}$, compared to 1.254×10^{-4} m s$^{-1/2}$ for wetting soil.

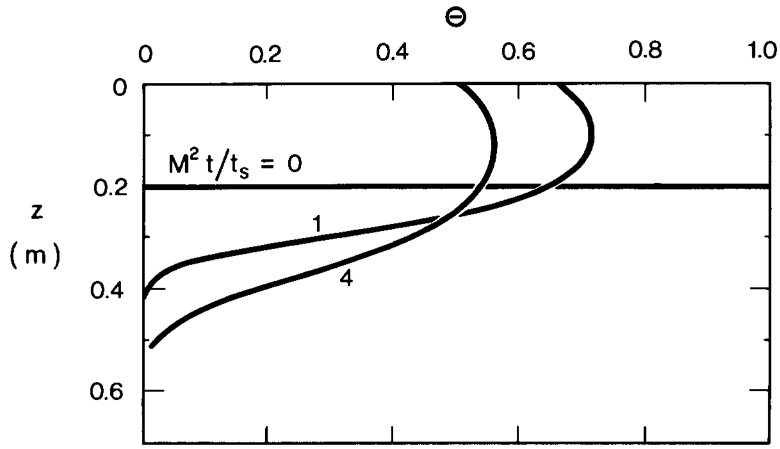

Fig. 3. Analytic solution for redistribution in Yolo light clay initially saturated to depth 0.2 m. Here, $M^{-2}t_s \simeq 14.4$ h. The moisture characteristic $\Psi(\theta)$ is taken to be the boundary-drying curve.

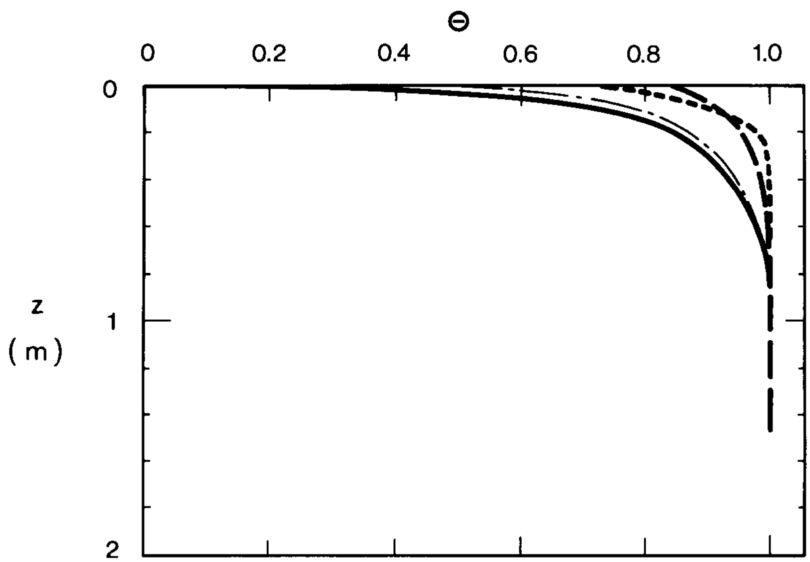

Fig. 4. Analytic solution for drainage of an initially saturated semi-infinite column of Yolo light clay at dimensionless time $M^2T = 1.0$. – – – current analytic model. · · · · Burgers model. —— hyperbolic model (Sisson et al. [31]). – · – · – improved hyperbolic model.

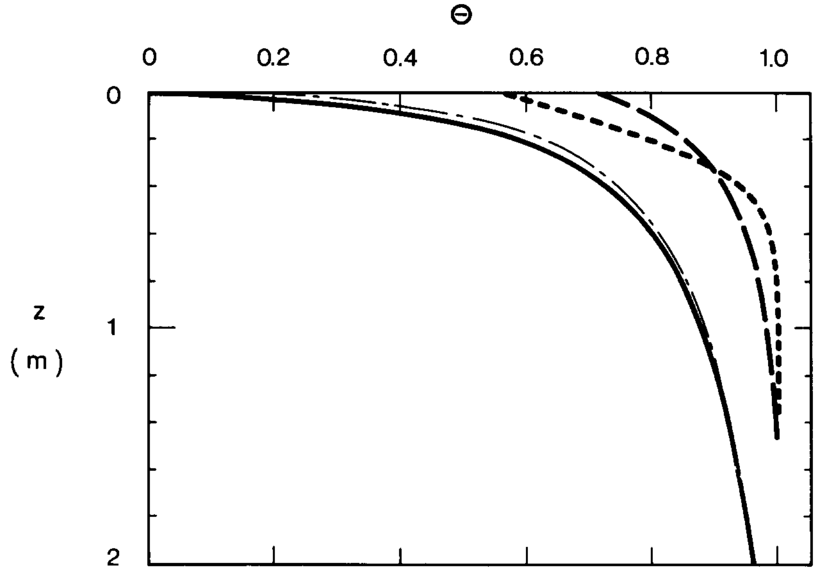

Fig. 5. Analytic solutions of Fig. 4 at dimensionless time $M^2T = 4.0$.

Values of the diffusivity (5) have been estimated by differentiating raw $\Psi(\theta)$ data and the irregularities in measured $D(\theta)$ may be a consequence of the noise in the input. The fitted diffusivity model (6b) seems to be a reasonably smooth representation of the raw data (Fig. 2). In Fig. 3 we plot the analytic solution for nonhysteretic redistribution of water in Yolo light clay initially saturated to depth 20 cm, displayed at dimensionless times $M^2T = 1.0$ and 4.0. Figures 4 and 5 include the solution for deep drainage of an initially saturated semi-infinite column.

3. Analytic solution (finite column)

Consider the nonhysteretic redistribution of moisture in a finite-length column with impermeable boundaries. The case of nonlinear diffusion with diffusivity of the form (6b) was solved by Knight and Philip [29]. Here we incorporate gravity by assuming a conductivity function of the form (6a), thereby correcting the deficiency recognised by Mahmoodian-Shoostari et al. [2].

Let us assume that the column has dimensionless depth z_*. From (21a), for any $z_{1*} \in [0, z_{0*}]$,

$$\int_0^{z_{1*}} \Theta \, dz_* = Cz_{1*} - \tfrac{1}{2}m^{1/2}Z(z_{1*}, t_*) . \tag{37}$$

Since the total volume V_0 of water in the column is constant, the lower boundary must correspond to a constant value $Z = Z_0$. From (37),

$$Z_0 = 2m^{-1/2}\left[Cz_{0*} - \frac{V_0 - V_n}{\lambda_s(\theta_s - \theta_n)A} \right], \tag{38}$$

where A is the cross-sectional area and $V_n = \lambda_s \theta_n z_{0*} A$.

The unsaturated-flow equation (8) is to be solved, subject to zero-flux boundary conditions

$$v_* = 0 \quad \text{at} \quad z_* = 0, z_{0*} \tag{39}$$

and general initial conditions

$$\Theta = \Theta_i(z_*) \quad \text{at} \quad t_* = 0\,, \tag{40}$$

As in the semi-infinite column, after applying the Kirchhoff, Storm and Cole–Hopf transformations consecutively, the flow equation transforms to the linear diffusion equation (27) while the boundary condition at $Z = Z_0$ must now be the same as that at $Z = 0$, namely

$$\frac{\partial^2 u}{\partial Z^2}\left(=\frac{\partial u}{\partial T}\right) = 0 \quad \text{at} \quad Z = 0, Z_0 \tag{41}$$

and the initial conditions are general,

$$u(Z) = u_0(Z) \quad \text{at} \quad T = 0\,, \quad \text{with } u_0 \text{ continuous}\,.$$

We define $f_0(Z)$ by:

$$f_0(Z) = u_0(0) + (Z/Z_0)[u_0(Z_0) - u_0(0)] \quad \text{(Fig. 6)}\,.$$

The function $g(Z, T) = u(Z, T) - f_0(Z)$ vanishes at $Z = 0$, Z_0 when $T = 0$ and by (41), this must be true at all times. We may extend $g(Z, T)$ skew-symmetrically at $Z = 0$, Z_0 and since $g(Z, T)$ vanishes at the boundaries, this skew-symmetric extension is continuous (Fig. 6). Therefore, we may expand $g(Z, T)$ as a Fourier sine series without any risk of incurring the Gibbs effect (e.g. Hamming [30]) when in practice the series is truncated.

$$u(Z, T) - f_0(Z) = g(Z, T) = \sum_{j=1}^{\infty} a_j(T) \sin \frac{\pi j Z}{Z_0} \tag{42a}$$

where

$$a_j(T) = \frac{2}{Z_0} \int_0^{Z_0} g(z, T) \sin \frac{\pi j Z}{Z_0}\, dZ\,. \tag{42b}$$

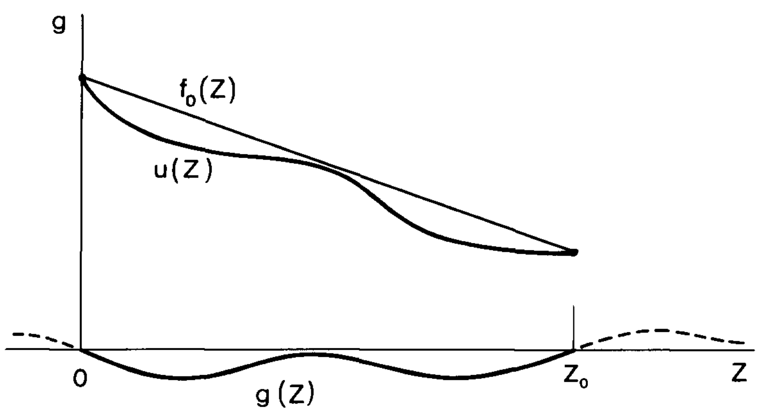

Fig. 6. Definition of $g(Z)$.

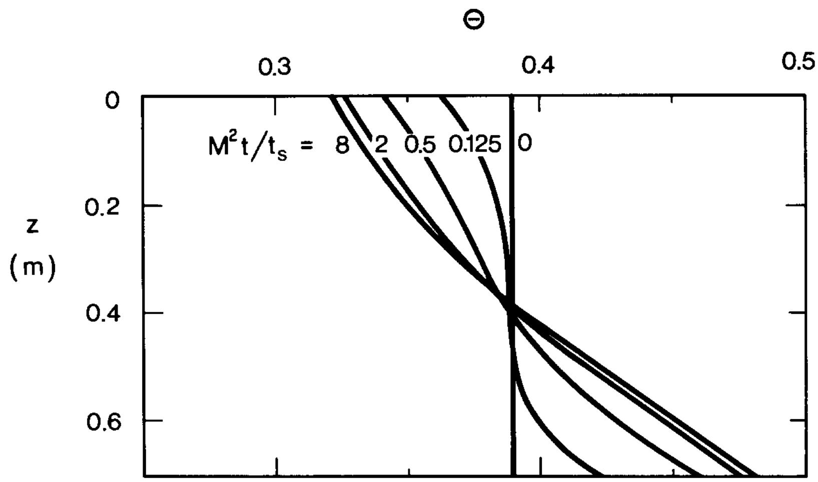

Fig. 7. Analytic solution for nonhysteretic redistribution in a 0.7 m vertical column of Yolo light clay initially at uniform volumetric moisture content $\theta = 0.388$. Here, $M^{-2}t_s \doteq 21$ days.

Now $u(Z, T)$ given in (42a) satisfies the linear diffusion equation (27), provided

$$\dot{a}_j(T) = -\left(\frac{\pi j}{Z_0}\right)^2 a_j ,$$

which implies

$$a_j(T) = a_j(0) \exp\left[-\left\{\frac{\pi j}{Z_0}\right\}^2 T\right] . \tag{43}$$

Given the initial conditions, $a_j(0)$ may be calculated and hence (43) provides a solution in the form of a Fourier series (42a) which is valid at all times.

To demonstrate the effect of gravity, we consider a finite column which initially has a uniform intermediate moisture content $\Theta = \Theta_i$ (constant). In the absence of gravity, the infiltration equation reduces to a nonlinear diffusion equation which would have the trivial solution $\Theta(z_*, t_*) = \Theta_i$, predicting no redistribution. The initial conditions here are the same as those considered for deep drainage except that Θ_i is now less than 1. By analogy with (29), the initial conditions must be

$$u = u_0 = e^{-MZ} \quad \text{at} \quad T = 0 . \tag{44}$$

Here,

$$M = [C/(C-1)]^{1/2}(\mu_i - C + 1) , \quad \text{with} \quad \mu_i = C(C-1)/(C - \Theta_i) . \tag{45}$$

The boundary condition (41) simplifies to

$$u = 1 \quad \text{at} \quad Z = 0 , \qquad u = e^{-MZ_0} \quad \text{at} \quad Z = Z_0 .$$

From (42b),

$$a_j(0) = \frac{2}{Z_0 \pi_j}(1 - e^{-MZ_0})(1-(-1)^j) + \frac{2}{\pi_j}[(-1)^j - 1]$$
$$+ \frac{2\pi_j}{Z_0^2}\left[M^2 + \left(\frac{\pi_j}{Z_0}\right)^2\right]^{-1}[1-(-1)^j e^{-MZ_0}]. \tag{46}$$

Then, from (42a) and (43),

$$u(Z, T) = 1 + \frac{Z}{Z_0}(e^{-MZ_0} - 1) + \sum_{j=1}^{\infty} a_j(0) \exp\left(-\left\{\frac{\pi j}{Z_0}\right\}^2 T\right) \sin \frac{\pi j Z}{Z_0}. \tag{47}$$

We may calculate Θ from (33), using

$$\frac{\partial u}{\partial Z} = Z_0^{-1}(e^{-MZ_0} - 1) + \sum_{j=1}^{\infty} \frac{\pi j}{Z_0} a_j(0) \exp\left(-\left\{\frac{\pi j}{Z_0}\right\}^2 T\right\} \cos \frac{\pi j Z}{Z_0}. \tag{48}$$

Equations (33)–(34) and (47)–(48) then provide an exact parametric solution. At large times the moisture profile approaches an equilibrium profile in which capillary forces balance gravitational forces, the gradient of total potential being zero. By Darcy's law, this means that the flux is zero everywhere,

$$0 = \frac{v}{K} = \left(1 - \frac{\partial \Psi}{\partial z}\right),$$

implying $\Psi = z + \Psi_0$, Ψ_0 being the asymptotic capillary potential at $z = 0$. It is well-known that in the absence of hysteresis, the profile $z(\theta)$ would resemble a portion of the moisture characteristic $\Psi(\theta)$ (e.g. Watson [6]).

From (26) and (47), the asymptotic moisture profile is given by

$$\mu = (C-1)\left\{1 + 2m^{-1/2} Z_0^{-1}(1 - e^{-MZ_0})\left[1 - \frac{Z}{Z_0}(1 - e^{-MZ_0})\right]^{-1}\right\}, \tag{49}$$

so that, at the surface $Z = 0$,

$$\mu \to \mu_\infty = (C-1)\{1 + 2m^{-1/2} Z_0^{-1}(1 - e^{-MZ_0})\}, \tag{50}$$

$$\Theta \to \Theta_\infty = C[1 - \{1 + 2m^{-1/2} Z_0^{-1}(1 - e^{-MZ_0})\}^{-1}], \tag{51}$$

and

$$\theta \to (\theta_s - \theta_n)\Theta_\infty + \theta_n. \tag{52}$$

The asymptotic moisture profile will then be given by a portion of the moisture characteristic beginning at this value of moisture.

In practice, the Fourier series (47)–(48) do not converge rapidly at early times. At dimensionless times $M^2 T$ less than one, it is more efficient to replace (47)–(48) by alternative series of complementary error functions, obtained by the method of Laplace transforms (Appendix).

Figure 7 shows the analytic solution for the redistribution of water in the same model soil depicted in Fig. 3a of White and Broadbridge [20] and Fig. 2 here, but this time in a vertical column of dimensionless length $z_{0*} = 4.775C^{-1}$ and with initial uniform moisture content $\Theta_i = \frac{1}{2}C$.

4. Comparison with simplified models

(*i*) *The hyperbolic model*

The replacement of equation (4) by the hyperbolic equation

$$\frac{\partial \theta}{\partial t} = -\frac{\partial K}{\partial z} \tag{53}$$

has been of some use in estimating the total amount of moisture

$$W(z_1, t) = \int_0^{z_1} \theta(z, t)\,\mathrm{d}z$$

down to moderate depths z_1 in a draining soil (e.g. Gardner et al. [12] and Nielsen et al. [1]). Equation (53) is associated with the so-called unit-gradient approximation, the capillary potential gradient $\partial\Psi/\partial z$ being assumed negligible compared to -1, the gravitational potential gradient. However, equation (53) also follows from (4) when one makes the weaker assumption that the gradient of the capillary component of moisture flux is small compared to the gradient of $K(\theta)$, the flux components themselves possibly being of the same order of magnitude. In fact, at $z = 0$, the zero-flux boundary condition of the redistribution problem requires that the downward gravitational component of flux should be balanced by upward (including diffusive) components so that the unit-gradient approximation cannot be valid near the surface. In this circumstance, it would seem unlikely for the weaker condition of negligible capillary-flux gradient to hold and we share, along with previous users of equation (53), a distrust of its predictions near $z = 0$.

Equation (53) can, in principle, always be solved by the method of characteristics (Sisson et al. [31]), since it follows that

$$\frac{\partial z(\theta, t)}{\partial t} = \frac{\mathrm{d}K}{\mathrm{d}\theta}, \tag{54}$$

implying that moisture contours $\theta = \theta_1$ propagate with speed $K'(\theta_1) = [\mathrm{d}K/\mathrm{d}\theta]_{\theta=\theta_1}$. Considering that θ is defined only on the ray $z \geqslant 0$, the most general solution which does not exhibit shock fronts is

$$\frac{z}{t - t_0(\theta)} = K'(\theta) \quad \text{for} \quad t \geqslant t_0(\theta), \tag{55}$$

with $t_0(\theta)$ an arbitrary monotonic decreasing function showing the time $t_0(\theta)$ at which the surface has moisture content θ. Figure 8 shows a typical set of characteristic curves $z(\theta, t)$ (θ fixed) corresponding to initial conditions $\theta = \theta_s$. It is clear that characteristics do not intersect and that shock fronts do not develop. If θ were defined on the whole of the real line $-\infty < z < \infty$, then the solution given by Lax [32] would be the only shock-free solution and it satisfies initial conditions

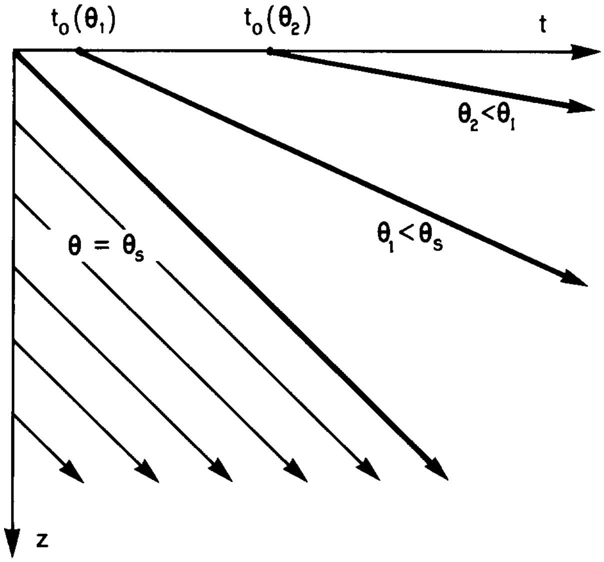

Fig. 8. Typical characteristic curves depicting propagation of moisture contours $\theta = \theta_l$, given initial saturation of semi-infinite region $z > 0$.

$$\theta = \begin{cases} \theta_s & (z > 0) \\ \theta_n & (z < 0)\,. \end{cases}$$

This was the only solution considered by Sisson et al. [31] and its restriction to $z > 0$ would correspond to (55) with $t_0(\theta) = 0$. In this solution, surface moisture drops immediately to the background level θ_n, as shown in Fig. 2 of Sisson et al. [31]. However, in our analytic solution, the appropriate time scale for surface drainage is $M^{-2}t_s$, which depends on the initial moisture content. In our example of initially saturated Yolo light clay, this time is approximately 14.4 h. The hyperbolic model may yet be improved by retaining the full boundary condition

$$0 = v = D(\theta)\,\frac{\partial \theta}{\partial z} - K(\theta) \quad \text{at} \quad z = 0 \tag{56}$$

when solving the approximate equation (54) exactly.

From equation (55),

$$\frac{\partial z(\theta, t)}{\partial \theta} = -\,t_0'(\theta)K'(\theta) + [t - t_0(\theta)]K''(\theta)\,.$$

However, at $z = 0$, $t = t_0(\theta)$ and by (56),

$$\frac{\partial z}{\partial \theta} = \frac{D(\theta)}{K(\theta)}\,.$$

Therefore, the boundary condition (56) is satisfied when

$$t_0'(\theta) = -\,\frac{D(\theta)}{K(\theta)K'(\theta)}\,. \tag{57}$$

For commonly assumed diffusivity and conductivity functions, (57) may be integrated exactly, leading to an analytic solution (55). Assuming that $D(\theta)$ and $K(\theta)$ have the form (6) of our current analytic model, we obtain

$$t_0(\theta) = m^{-1}t_s\left[\ln\left(\left[\frac{(2C/\Theta)-1}{2C-1}\right]^{1/2}\right) + C\left(\frac{1}{2C-1} - \frac{1}{2C-\Theta}\right) + C^2\left(\frac{1}{(2C-\Theta)^2} - \frac{1}{(2C-1)^2}\right)\right]. \qquad (58)$$

At times large compared to $m^{-1}t_s$, the improved hyperbolic model agrees closely with the solution of Sisson et al. [31], as shown in Fig. 5. At early times, the improved hyperbolic model still underestimates the surface water content but it predicts a value significantly higher than θ_n (Fig. 4).

(ii) The Burgers equation

Analytic solutions to constant-flux infiltration have previously been obtained by assuming that $K(\theta)$ is quadratic and that D is constant (Clothier et al. [33]). The unsaturated flow equation (4) then reduces to Burgers' equation

$$\frac{\partial\Theta}{\partial t} = D\frac{\partial^2\Theta}{\partial z^2} - 2U\Theta\frac{\partial\Theta}{\partial z}, \quad \text{with} \quad U = K_s/(\theta_s - \theta_n). \qquad (59)$$

Although this model has not previously been used in redistribution studies, it may be considered to be a more accurate representation than the hyperbolic equation (54), which completely neglects diffusion. For deep drainage, the initial condition is $\Theta = 1$ and the boundary condition is

$$0 = U\Theta^2 - D\frac{\partial\Theta}{\partial z} \quad \text{at} \quad z = 0. \qquad (60)$$

This problem is virtually identical to that of equations (22)–(24), except that now z is ordinary depth rather than material coordinates Z and Θ is normalized volumetric water content rather than Kirchhoff variable μ. By the procedure which we have already used to solve equation (22), we obtain

$$\Theta = \frac{-D}{U}u^{-1}\frac{\partial u}{\partial z}, \qquad (61)$$

where

$$u = \exp(M^2t - Mz) + \operatorname{erfc}(\tfrac{1}{2}zt^{-1/2}) - \tfrac{1}{2}e^{M^2t}\{e^{-Mz}\operatorname{erfc}[\tfrac{1}{2}zt^{-1/2} - Mt^{1/2}] + e^{Mz}\operatorname{erfc}[\tfrac{1}{2}zt^{-1/2} + Mt^{1/2}]\} \qquad (62)$$

and

$$\partial u/\partial z = -M\exp(M^2t - Mz) - \tfrac{1}{2}M\exp(M^2t + Mz)\operatorname{erfc}[\tfrac{1}{2}zt^{-1/2} + Mt^{1/2}] + \tfrac{1}{2}M\exp(M^2t - Mz)\operatorname{erfc}[\tfrac{1}{2}zt^{-1/2} - Mt^{1/2}]. \qquad (63)$$

In this case, $M = U/D$. We choose the standard representative diffusivity of Philip [15],

$$D = \tfrac{1}{4}\pi S^2/(\theta_s - \theta_n)^2 ,$$

with S representative of drying soil. The solution of the Burgers model is plotted in Figs. 4 and 5, along with solutions of the hyperbolic and full analytic models for Yolo light clay. The hyperbolic model, implying maximum propagation speed of $K'(\theta_s)$, leads to a discontinuity in $\partial\theta/\partial z$ at depth $tK'(\theta_s)$. This discontinuity may be viewed as a singularity in the second derivative $\partial^2\theta/\partial z^2$. Regions of high curvature in the moisture profile will be smoothed when diffusion is incorporated. However, the representative diffusivity used in the Burgers model is low compared to $D(\theta_s)$ and hence regions of high curvature in $\theta(z)$ are more noticeable in the Burgers model than in the more realistic current analytic model. It is fair to say that in the example of Figs. 4 and 5, moisture profiles predicted by the Burgers model are more accurate than those predicted by the hyperbolic model.

Acknowledgement

We gratefully acknowledge the contribution of Dr. J.H. Knight who collaborated with us throughout our investigation.

Appendix

We apply the Laplace transformation to solve the linear diffusion equation (27) subject to boundary conditions (28) and initial conditions (29), which arise in the theory of redistribution in a semi-infinite column. Secondly, by similar techniques, we solve the diffusion equation subject to boundary conditions (41) and initial conditions (44), which arise in the theory of redistribution in a finite column.

Let $v = u - \exp(-ML)$ and $\bar{v}(Z, p)$ be the Laplace transform of $v(Z, T)$. Since v satisfies the diffusion equation (27), $\bar{v}$ satisfies

$$\frac{d^2\bar{v}}{dZ^2} - p\bar{v} = -v_0(Z) , \tag{A1}$$

where $v_0(Z)$ is the initial condition (e.g. Ch. 12 of Carslaw and Jaeger [34]). From the initial conditions (29),

$$v_0(Z) = \begin{cases} e^{-MZ} - e^{-ML} , & Z \leq L , \\ 0 , & Z \geq L . \end{cases} \tag{A2}$$

Let $\bar{v} = A(Z, p)\exp(-p^{1/2}Z)$, and $B(Z, p) = d/dZ\, A(Z, p)$. Then (A1) implies

$$e^{-p^{1/2}Z}\, dB/dZ - 2p^{1/2}B\, e^{-p^{1/2}Z} = -v_0(Z) . \tag{A3}$$

Let $B = C(Z, p)\, e^{2p^{1/2}Z}$. Then (A3) implies

$$\frac{dC}{dZ} = -e^{-p^{1/2}Z}\, v_0(Z) . \tag{A4}$$

Using (A2), (A4) has the solution

$$C(Z, p) = \frac{1}{p^{1/2} + M} e^{-(p^{1/2}+M)Z} - \frac{e^{-ML}}{p^{1/2}} e^{-p^{1/2}Z} + r(p), \quad \text{if} \quad Z \leqslant L,$$
$$= \frac{1}{p^{1/2} + M} e^{-(p^{1/2}+M)Z} - \frac{e^{-ML}}{p^{1/2}} e^{-p^{1/2}L} + r(p), \quad \text{if} \quad Z \geqslant L, \tag{A5}$$

with $r(p)$ some function of p alone. From C, we deduce B and A and it follows that

$$\bar{v} = \frac{e^{-MZ}}{p - M^2} - \frac{e^{-ML}}{p} + q(p)\, e^{-p^{1/2}Z} + \frac{r(p)}{2p^{1/2}} e^{p^{1/2}Z}, \quad \text{if} \quad Z \leqslant L,$$
$$= \frac{1}{2}\left\{\frac{1}{p^{1/2}(p^{1/2} + M)} e^{-(p^{1/2}+M)L} - \frac{e^{-ML}}{p} e^{-p^{1/2}L}\right\} e^{p^{1/2}Z} + m(p)\, e^{-p^{1/2}Z}$$
$$+ \frac{r(p)}{2p^{1/2}} e^{p^{1/2}Z}, \quad \text{if} \quad Z \geqslant L. \tag{A6}$$

where $q(p)$ and $m(p)$ are functions of p alone. The functions $m(p)$, $q(p)$ and $r(p)$ may be determined by the boundary conditions $d^2\bar{v}/dZ^2 = 0$ at $Z = 0$, ∞ and by the condition that $\bar{v}$ is continuous at $Z = L$. The final expression for $\bar{v}$ is

$$\bar{v} = \frac{e^{-MZ}}{p - M^2} - \frac{e^{-ML}}{p} + \tfrac{1}{2} e^{-ML} \frac{e^{-p^{1/2}}(L + Z)}{p^{1/2}(p^{1/2} + M)}$$
$$- \tfrac{1}{2} e^{-ML} \frac{e^{-p^{1/2}(L+Z)}}{p} - M^2 \frac{e^{-p^{1/2}Z}}{p(p - M^2)}$$
$$- \tfrac{1}{2} e^{-ML}\left\{\frac{e^{p^{1/2}(Z-L)}}{p^{1/2}(p^{1/2} + M)} - \frac{e^{-p^{1/2}(L-Z)}}{p}\right\}, \quad \text{if} \quad Z \leqslant L,$$
$$= -M^2 \frac{e^{-p^{1/2}Z}}{p(p - M^2)} + \tfrac{1}{2} e^{-ML}\left\{\frac{e^{-p^{1/2}(Z+L)}}{p^{1/2}(p^{1/2} + M)} - \frac{e^{-p^{1/2}(Z-L)}}{p^{1/2}(p^{1/2} + M)}\right.$$
$$\left. - \frac{e^{-p^{1/2}(L+Z)}}{p} - \frac{e^{p^{1/2}(L-Z)}}{p} + 2\frac{e^{-p^{1/2}(Z-L)}}{p - M^2}\right\}, \quad \text{if} \quad Z \geqslant L. \tag{A7}$$

We may expand rational functions of p as partial fractions. For example, we simplify (A7) by using

$$\frac{-M^2}{p(p - M^2)} = \frac{1}{p} - \frac{1}{p - M^2}. \tag{A8}$$

Then v may be obtained from $\bar{v}$ using tables of standard inverse Laplace transforms (e.g. Appendix V of Carslaw and Jaeger [34]) and the expressions (30a, b) follow.

For the case of redistribution in a finite column, the Laplace transform $\bar{u}$ of u must satisfy

$$\frac{d^2\bar{u}}{dZ^2} - p\bar{u} = -u_0(Z) = -e^{-MZ}. \tag{A9}$$

By the methods used for the infinite column, the solution is

$$\bar{u} = \frac{e^{-MZ}}{p - M^2} + \tfrac{1}{2}E(p)\frac{e^{p^{1/2}Z}}{p^{1/2}} + F(p)\,e^{-p^{1/2}Z}\,. \tag{A10}$$

The functions $E(p)$ and $F(p)$ may be determined by the boundary conditions $d^2\bar{u}/dZ^2 = 0$ at $Z = 0$, Z_0, with the result

$$\begin{aligned}\bar{u} = {} & \frac{e^{-MZ}}{p - M^2} - \frac{M^2}{p(p - M^2)}\,e^{-p^{1/2}Z} \\ & + \frac{M^2}{p(p - M^2)}\,e^{-p^{1/2}Z_0}\,\frac{\sinh(p^{1/2}Z)}{\sinh(p^{1/2}Z_0)} \\ & - \frac{M^2}{p(p - M^2)}\,e^{-MZ_0}\,\frac{\sinh(p^{1/2}Z)}{\sinh(p^{1/2}Z_0)}\,.\end{aligned} \tag{A11}$$

Once again, we use (A8) and following the treatment of a similar problem (Sec. 12.5 of Carslaw and Jaeger [34]), we expand $[\sinh(p^{1/2}Z_0)]^{-1}$ as

$$\frac{2}{e^{p^{1/2}Z_0}(1 - e^{-2p^{1/2}Z_0})} = 2\sum_{n=0}^{\infty} e^{-(2n+1)p^{1/2}Z_0}\,. \tag{A12}$$

Application of the inverse Laplace transform to (A11) then yields

$$\begin{aligned}u = {} & e^{-MZ+M^2T} + \operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2}) \\ & - \tfrac{1}{2}\,e^{M^2T}\{e^{-MZ}\operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2} - MT^{1/2}) + e^{MZ}\operatorname{erfc}(\tfrac{1}{2}ZT^{-1/2} + MT^{1/2})\} \\ & + \sum_{n=0}^{\infty} \tfrac{1}{2}\,e^{M^2T}\{e^{-M(2Z_0-Z+2Z_0n)}\operatorname{erfc}(\tfrac{1}{2}[2Z_0 - Z + 2Z_0n]T^{-1/2} - MT^{1/2}) \\ & + e^{M(2Z_0-Z+2Z_0n)}\operatorname{erfc}(\tfrac{1}{2}[2Z_0 - Z + 2Z_0n)T^{-1/2} + MT^{1/2})\} \\ & - \tfrac{1}{2}\,e^{M^2T}\{e^{-M(2Z_0+Z+2Z_0n)}\operatorname{erfc}(\tfrac{1}{2}[2Z_0 + Z + 2Z_0n]T^{-1/2} - MT^{1/2}) \\ & + e^{M(2Z_0+Z+2Z_0n)}\operatorname{erfc}(\tfrac{1}{2}[2Z_0 + Z + 2Z_0n)T^{-1/2} + MT^{1/2})\} \\ & + \tfrac{1}{2}\,e^{-MZ_0+M^2T}\{e^{-M(Z_0+Z+2Z_0n)}\operatorname{erfc}(\tfrac{1}{2}[Z_0 + Z + 2Z_0n]T^{-1/2} - MT^{1/2}) \\ & + e^{M(Z_0+Z+2Z_0n)}\operatorname{erfc}(\tfrac{1}{2}[Z_0 + Z + 2Z_0n)T^{-1/2} + MT^{1/2})\} \\ & - \tfrac{1}{2}\,e^{-MZ_0+M^2T}\{e^{-M(Z_0-Z+2Z_0n)}\operatorname{erfc}(\tfrac{1}{2}[Z_0 - Z + 2Z_0n]T^{-1/2} - MT^{1/2}) \\ & + e^{M(Z_0-Z+2Z_0n)}\operatorname{erfc}(\tfrac{1}{2}[Z_0 - Z + 2Z_0n]T^{-1/2} + MT^{1/2})\} \\ & - \operatorname{erfc}(\tfrac{1}{2}[2Z_0 - Z + 2Z_0n]T^{-1/2}) + e^{-MZ_0}\operatorname{erfc}(\tfrac{1}{2}[Z_0 - Z + 2Z_0n]T^{-1/2}) \\ & + \operatorname{erfc}(\tfrac{1}{2}[2Z_0 + Z + 2Z_0n]T^{-1/2}) - e^{-MZ_0}\operatorname{erfc}(\tfrac{1}{2}[Z_0 + Z + 2Z_0n]T^{-1/2})\,.\end{aligned} \tag{A13}$$

The above series converges rapidly for small t and therefore it is convenient to use it at dimensionless times $M^2T < 1$.

References

1. D.R. Nielsen, J.W. Biggar and K.T. Erh, Spatial variability of field-measured soil-water properties, *Hilgardia* 42 (1973) 215–259.
2. M. Mahmoodian-Shooshtari, L.A. Davis, D.M. McWhorter and A. Klute, Unsaturated soil hydraulic properties from redistribution of injected water, *Trans. ASAE* 27 (1984) 784–790.
3. A.J. Jones and R.J. Wagenet, *In situ* estimation of hydraulic conductivity using simplified methods, *Water Resour. Res.* 20 (1984) 1620–26.
4. E.C. Childs, The ultimate moisture profile during infiltration in a uniform soil, *Soil Sci.* 97 (1964) 173–178.
5. E.G. Youngs and A. Poulovassilis, The different forms of moisture-profile development during the redistribution of soil water after infiltration, *Water Resour. Res.* 12 (1976) 1007–12.
6. K.K. Watson and V. Sardana, Numerical study of the effect of hysteresis on post-infiltration redistribution, in *Infiltration Development and Application*, (Ed. Y.-S. Fok), Water Resources Research Center, Honolulu (1987).
7. F.D. Whisler and A. Klute, The numerical analysis of infiltration, considering hysteresis, into a vertical soil column at equilibrium under gravity, *Soil Sci. Soc. Am. Proc.* 29 (1965) 489–494.
8. J. Rubin, Numerical method for analysing hysteresis-affected, post-infiltration redistribution of soil moisture, *Soil Sci. Soc. Am. Proc.* 31 (1967) 13–20.
9. W.J. Staple, Comparison of computed and measured moisture redistribution following infiltration, *Soil Sci. Soc. Am. Proc.* 33 (1969) 840–847.
10. A.J. Peck, Redistribution of soil water after infiltration, *Aust. J. Soil Res.* 9 (1971) 59–71.
11. S.J. Perrens and K.K. Watson, Numerical analysis of 2D infiltration and redistribution, *Water Resour. Res.* 13 (1977) 781–790.
12. W.R. Gardner, D. Hillel and Y. Benyamini, Post-irrigation movement of soil water; 1. Redistribution, *Water Resour. Res.* 6 (1970) 851–861.
13. T. Talsma, The effect of initial moisture content and infiltration quantity on redistribution of soil water, *Aust. J. Soil Res.* 12 (1974) 15–26.
14. P.A.C. Raats, Analytic solution of a simplified flow equation, *Trans. ASAE* 19 (1976) 683–689.
15. J.R. Philip, Theory of infiltration, *Adv. Hyrosci.* 5 (1969) 215–296.
16. A.S. Fokas and Y.C. Yortsos, On the exactly solvable equation occurring in two-phase flow in porous media, *SIAM J. Appl. Math.* 42 (1982) 318–332.
17. G. Rosen, Method for the exact solution of a nonlinear diffusion-convection equation, *Phys. Rev. Lett.* 49 (1982) 1844–46.
18. C. Rogers, M.P. Stallybrass and D.L. Clements, On two phase filtration under gravity and with boundary infiltration: application of a Bäcklund transformation, *J. Nonlinear Anal.* 7 (1983) 785–799.
19. P. Broadbridge and I. White, Constant-rate rainfall infiltration: a versatile nonlinear model; I. Analytic solution, *Water Resour. Res.* 24 (1988) 145–154.
20. I. White and P. Broadbridge, Constant-rate rainfall infiltration: a versatile nonlinear model; II. Applications of solutions, *Water Resour. Res.* 24 (1988) 155–162.
21. J.R. Philip, The theory of infiltration: 4. Sorptivity and algebraic infiltration equations, *Soil Sci.* 84 (1957) 257–264.
22. G. Kirchhoff, *Vorlesungen über die Theorie der Wärme*, Barth, Leipzig (1894).
23. M.L. Storm, Heat conduction in simple metals, *J. Appl. Phys.* 22 (1951) 940–951.
24. A.R. Forsyth, *Theory of Differential Equations*, vol. 6, Dover publications, New York (1959). (Previously published in 1906).
25. E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Commun. Pure Appl. Math.* 3 (1950) 201–230.
26. J.D. Cole, On a quasilinear parabolic equation occurring in aerodynamics, *Quart. Appl. Math.* 9 (1951) 225–236.
27. R.E. Moore, Water conduction from shallow water tables, *Hilgardia* 12 (1939) 383–426.
28. J.R. Philip and J.H. Knight, On solving the unsaturated flow equation; 3. New quasi-analytical technique. *Soil Sci.* 117 (1974) 1–13.
29. J.H. Knight and J.R. Philip, Exact solutions in nonlinear diffusion, *J. Engng. Math.* 8 (1974) 219–227.
30. R.W. Hamming, *Numerical Methods for Scientists and Engineers*, McGraw-Hill, New York (1962).
31. J.B. Sisson, A.H. Ferguson and M.Th. van Genuchten, Simple method for predicting drainage from field plots, *Soil Sci. Soc. Am. J.* 44 (1980) 1147–52.
32. P.D. Lax, The formation and decay of shock waves, *Am. Math. Monthly* 79 (1972) 227–241.
33. B.E. Clothier, J.H. Knight and I. White, Burgers' equation: application to field constant-flux infiltration, *Soil Sci.* 132 (1981) 255–261.
34. H.S. Carslaw and J.C. Jaeger, *Conduction of Heat in Solids*, 2nd edition, Clarendon Press, Oxford (1959).